Weighted scale-free networks in Euclidean space using local selection rule

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A spatial scale-free network is introduced and studied, whose motivation originated in the growing Internet as well as airport networks. We argue that in these real-world networks a new node necessarily selects one of its neighboring local nodes for connection and is not controlled by preferential attachment as in the Barabási-Albert (BA) model. This observation is mimicked in our model where the nodes pop up at randomly located positions in the Euclidean space and are connected to one end of the nearest link. In spite of this crucial difference it is observed that the leading behavior of our network is like that of the BA model. Defining the link weight as an algebraic power of its Euclidean length, the weight distribution and the nonlinear dependence of the nodal strength on the degree are analytically calculated. It is claimed that a power law decay of the link weights with time ensures such nonlinear behavior. Switching off the Euclidean space from the same model yields a much simpler definition of the BA model where numerical effort grows linearly with *N*.

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Studying the structure of communication networks is important in its own right because it helps in understanding the network and its traffic flow distribution, which in turn helps in making the communication process more efficient. Over the last several years many real-world networks are being studied with much interest. The examples range over social networks, electronic networks, and biological networks [1–3]. An important subclass of these networks is spatial networks, i.e., those embedded in the Euclidean space. Two most important examples of these networks are electronic communication networks like the Internet [4–6], which is a transport network of electronic data packets, as well as the public transport system of airport networks [7,8].

The common property of these two networks is their highly inhomogeneous structure. This inhomogeneity is reflected in their nodal degree distribution (the degree k of a node is the number of links attached to it) which is observed to follow a power law distribution: $P(k) \sim k^{-\gamma_k}$. It was Barabási and Albert (BA) who first recognized that indeed there are many other real-world examples of social and biological networks having similar structures [9]. Since these networks lack a characteristic value for the nodal degree, they are called scale-free networks [1,9].

BA observed two key characteristics for these networks: (i) they are growing networks and (ii) an inherent "rich get richer" mechanism exists, which ensures that large-degree nodes grow at higher rates. In the BA model a network grows by addition of new nodes which become connected to the growing network using a linear attachment probability. In this paper we question the general necessity of this "linear attachment" rule. We argue that at least for spatial networks like the Internet and the airport network the existence of such a rule for the expanding network seems to be highly implausible.

If you buy a new computer and would like to connect it to the Internet what do you do? If it is a home computer you use a modem and a dial-up telephone line to connect it to the nearest router of the telephone company. If it is your office computer, it gets a connection to the office router, which is eventually connected to the nearest node of the Internet service provider (ISP) your organization opted for. If your office is a part of your university or organization, several routers are used and at the autonomous systems level they are connected to the nearest ISP again. Therefore new nodes of the Internet at any level are always connected to the existing local nodes of the network. A person in Chile would hardly give any extra importance to large ISP hubs in Tokyo, Stuttgart, or Chicago rather than small providers in his/her locality. Considering the whole world-wide Internet network it is apparent that new nodes pop up randomly in space and time without any spatial correlations.

Similar arguments can be put forward for the airport network as well. A new airport in some remote city in some country is quite likely to be connected to a neighboring airport first by direct nonstop flights and very unlikely to have cross-country intercontinental flights to begin with. Recent studies of International Air Transport Association airport network data also revealed that the weights w_{ij} of the links in this network defined either by the number of passenger seats or by the Euclidean distance between successive stops have nontrivial variations [10]. The nodal strength defined as $s_i = \Sigma_j w_{ij}$ varies as $\langle s(k)\rangle \propto k^{\beta}$ with $\beta > 1$. A number of models have been proposed to explain such nonlinearity [11–13].

In this paper we study a growing network on Euclidean space where new nodes are added one by one and are connected to the neighboring nodes of the growing network. We show that even such a network has a scale-free degree distribution. In addition, these networks have nontrivial dependence of the average strength on the nodal degree.

A spatial network is grown on a two-dimensional space whose nodes are points at randomly selected positions within a unit square on the x-y plane (with periodic boundary conditions) [14]. Let $\{(x_1,y_1),(x_2,y_2),\ldots,(x_N,y_N)\}$ represent the coordinates of N randomly distributed points within this space where each coordinate is an independent and identi-

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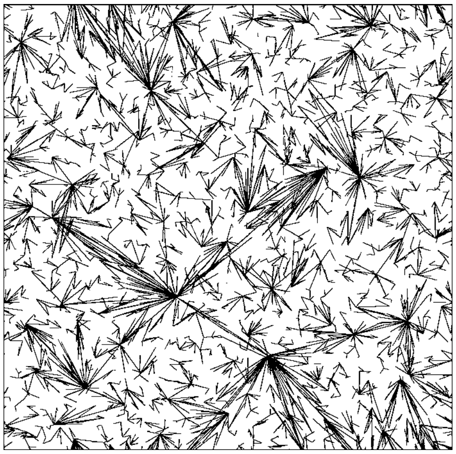

FIG. 1. The picture of a network of $N=2^{12}$ nodes; each node is a randomly positioned point on the unit square and is connected randomly to one end of the nearest link.

cally distributed number drawn within {0,1}. The growth of the network starts with a pair of nodes 1 and 2 connected by a single link. Nodes labeled 3 to N are then introduced one by one and are connected to the growing network.

At some intermediate stage let the network have t links and $t+1$ nodes. For connecting the $(t+2)$th node to the network, the nearest link center is selected. One of the two end nodes of the nearest link is then chosen with equal probability and is connected to the new node to create the $(t+1)$th link (Fig. 1). This is executed by keeping the local information into memory. The unit square area is divided into a lattice of size $\sqrt{N}\times\sqrt{N}$. As the network grows the serial numbers of the links whose centers are positioned within a lattice cell are stored at the associated lattice site. To find the nearest link center one starts from the cell of the $(t+2)$th node and searches the lattice cells shell by shell until the nearest link center is found out. When $t\sim N$ only the nearest shell needs to be searched.

After the network has grown to N nodes, the degree distribution $P(k,N)$ is calculated. From a direct measurement of the slope of the $\log P(k,N)$ vs $\log k$ plot the exponent γ_k is estimated to be 3.00 ± 0.05, which is close to the BA value. Moreover, a scaling of $P(k,N)$ is also studied [Fig. 2(a)]:

$$P(k,N) \propto N^{-\eta}\mathcal{G}(k/N^{\zeta}) \qquad (1)$$

where $\eta=3/2$ and $\zeta=1/2$ are used to obtain the best data collapse giving $\gamma_k=\eta/\zeta=3$. The network so generated has a tree structure. However, networks having more general structures with multiple loops can be generated with m links coming out from every incoming node and are attached in a similar way to the first m neighboring links in increasing order. In Fig. 2(b) we show that degree distributions for $m=2$ and 3 can be scaled as $P(k,N)m^{-1.67}\propto N^{-\eta}\mathcal{G}(k/N^{\zeta})$.

The link lengths are also measured and their probability distribution is calculated as studied in [14]. Let $\mathcal{D}(\ell)d\ell$ denote the probability that a randomly selected link has a length between ℓ and $\ell+d\ell$. For a given Poisson distribution of N points let us first calculate the first-neighbor-distance distribution. Consider a point P at an arbitrary position. The

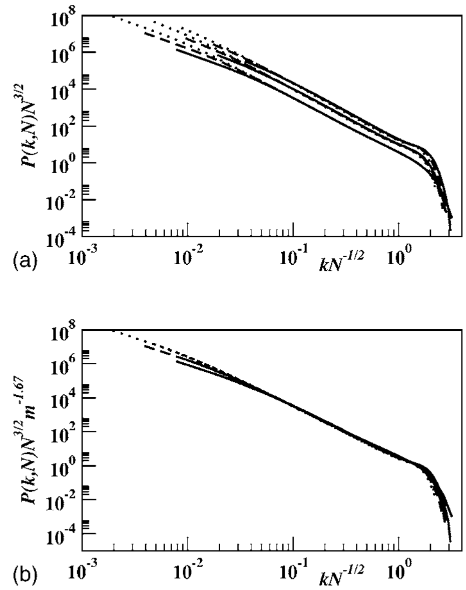

FIG. 2. (a) Finite-size scaling analysis of the degree distribution for three system sizes $N=2^{14}$ (solid line), 2^{16} (dashed line), and 2^{18} (dotted line) and for three values of the number of outgoing links $m=1$, 2, and 3 (from left to right). (b) Same plot as in (a) but scaled with $m^{-1.67}$.

probability that its first neighbor is situated at a distance within r and $r+dr$ (which can be done in $N-1$ different ways) and all other $N-2$ points are at distances greater than $r+dr$ is

$$P(r)dr = (N-1)2\pi r\, dr(1-\pi r^2)^{N-2}. \qquad (2)$$

In the limit of $N\to\infty$ it can be approximated that $N-1\approx N-2\approx N$ and since the average area per point decreases as $1/N$, πr^2 is very small compared to 1. Therefore $(1-\pi r^2)^{N-2}$ is approximated as $\exp(-\pi N r^2)$. Therefore in the limit of $N\to\infty$ the probability density distribution is $P(r)\approx 2\pi N r\exp(-\pi N r^2)$ or in the scaling form

$$P(r)/\sqrt{N} \propto (r\sqrt{N})\exp[-\pi(r\sqrt{N})^2] \qquad (3)$$

where the scaling length $1/\sqrt{N}$ is the linear extent of the average area per node.

In our case the number of nodes N in the system is not fixed; rather it grows with time. Therefore the link lengths ℓ also decrease as time progresses. After some time the total collection of links has a mixture of many different lengths. Since initially there were only a few nodes the early links are very large and may be as big as the box size, whereas the last few links are very small and have lengths of the order of $\ell_0\sim 1/\sqrt{N}$. The effect of the mixing is twofold as observed numerically. The distribution of small link lengths (later stage) up to $\ell_m\approx 2.5\ell_0$ is different from Eq. (3) but still follows a scaling form:

$$\mathcal{D}(\ell,N) \propto \sqrt{N}\mathcal{F}(\ell\sqrt{N}). \qquad (4)$$

The scaling function fits very well the generalized Gamma distribution $\mathcal{G}(x)=Ax^a\exp(-bx^c)$ where $A\approx 12.6$, $a\approx 1.4$, $b\approx 3.11$, and $c\approx 0.83$ [Fig. 3(a)]. On the other hand the long

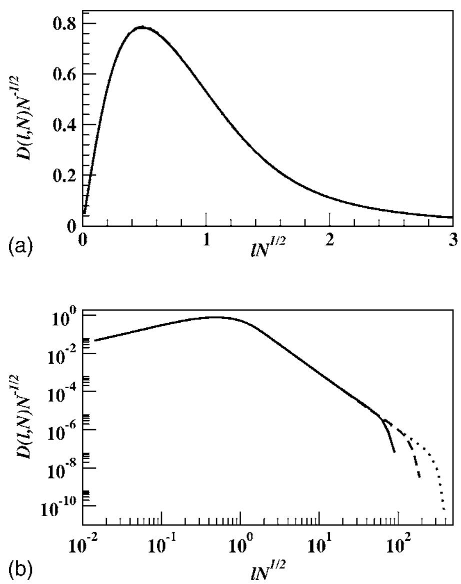

FIG. 3. Finite-size scaling analysis of the link length distribution $D(\ell,N)$ for three system sizes $N=2^{14}$ (solid line), 2^{16} (dashed line), and 2^{18} (dotted line). (a) shows the plots on a linear scale whereas in (b) a double-logarithmic scale has been used for the same data but binned on the exponential scale.

early links contribute to a power law tail as $\mathcal{D}(\ell)\sim\ell^{-\gamma_w}$ for $\ell_m<\ell<\sqrt{2}/2$ with $\gamma_w\approx 3.0$ [Fig. 3(b)].

Like the weighted airport network [10] the strength s_i of a node i is measured as the sum of the Euclidean lengths of all links meeting at i: $s_i=\Sigma_j\ell_{ij}^{\alpha}$, ℓ_{ij} being the length of the link between the nodes i and j and α a continuously tunable parameter. This parameter generalizes the model to take care of situations where the link weight may even vary nonlinearly with the Euclidean distance, e.g., the route that an aircraft takes for flying between two airports is quite often greater than the length of the geodesic path between them. For example, the ASIANA airline flies from Seoul to New Delhi through the air space of Bangladesh. Also in the mobile *ad hoc* network the power P_i required to maintain the range R_i of transmission of each mobile element varies nonlinearly with the range, $P_i\propto R_i^{\alpha}$, with $1<\alpha<6$.

We first observe that, given a connected network of t links, the whole space is partitioned into t nonoverlapping Voronoi cells, each cell surrounding the center of a link [15] (Fig. 4). The center of the link is at a minimum distance from all points within this cell. The probability that a randomly selected point is within a particular cell is equal to the area of the cell. Since different Voronoi cells have different areas the probability of selecting a link center is in general nonuniform. For a two-dimensional Poisson Voronoi tessellation the cell sizes follow a Gamma distribution whose width scales as $1/t$ [16]. Therefore though for finite t the cell sizes are nonuniform, for $t\to\infty$ the cell size distribution is similar to a δ function implying that all cells have uniform size. In this limit the probability that a particular node of degree k will be linked is $k/2$ times the cell size—which gives rise to linear preferential attachment as in the BA model. Therefore for a very large network ($N\to\infty$) in our model the node selection probability is different from that in the BA model at early

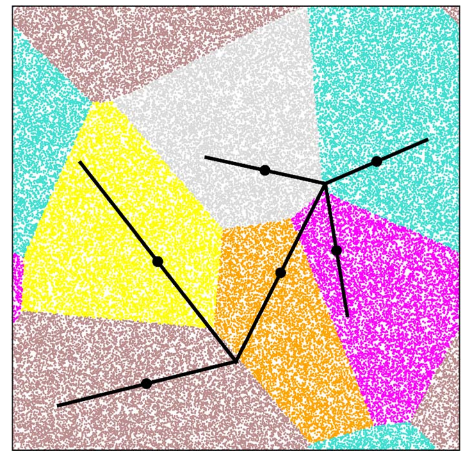

FIG. 4. (Color online) On a unit square a six-link network is drawn and the centers of the links are marked. The whole space is then partitioned into six different Voronoi cells around these centers. Cells are marked by dropping 2^{16} randomly selected points in this space and coloring with a particular color all points in a cell that are nearest to the corresponding link center.

times (t small) whereas it asymptotically converges to the BA preferential rule in the limit of $t\to\infty$. We conclude that the leading behavior of our spatial network model is like the BA model. We have already seen that even for finite N the degree exponent $\gamma_k\approx 3.0$ as in the BA model.

Let us assume that the area of every Voronoi cell in the network with t links is uniform and is equal to $1/t$. This implies that the ith node with degree $k_i(t)$ has probability $k_i(t)/(2t)$ to be linked with the new $(t+2)$th node. The factor 2 comes from the fact that one of the two end nodes of every link is selected with equal probability. Therefore $dk_i(t)/dt=k_i(t)/(2t)$ and $k_i(t)=(t/i)^{1/2}$, exactly similar to the BA model, resulting in $P(k)\propto k^{-3}$.

Again, since the area of each cell is $1/t$ the typical length ℓ_{ij} of the $(t+1)$th link to be connected to the node i is proportional to $(k_i(t)/t)^{1/2}$. Therefore the rate of increase of the strength of the ith node is

$$ds_i(t)/dt=[dk_i(t)/dt](\ell_{ij}^{\alpha})\propto[k_i(t)/t]^{1+\alpha/2}, \quad (5)$$

which is proportional to $(ti)^{-1/2-\alpha/4}$. On integration over t from $t=i$ to t,

$$s_i(t)-s_i(i)\propto\left(\frac{t^{1/2-\alpha/4}}{i^{1/2+\alpha/4}}-i^{-\alpha/2}\right). \quad (6)$$

The value of $s_i(i)$ is estimated by the average strength of the ith node when it was introduced and connected to an arbitrary previous node j, $j=1,i-1$,

$$s_i(i)\propto i^{-1/2-\alpha/4}. \quad (7)$$

When t is large in Eq. (6) the term $i^{-\alpha/2}$ is ignored,

$$s_i(t)\propto i^{-1/2-\alpha/4}(t^{1/2-\alpha/4}+c). \quad (8)$$

Writing i in terms of $s_i(t)$

$$i\propto(t^{1/2-\alpha/4}+c)^{4/(2+\alpha)}[s_i(t)]^{-4/(2+\alpha)}. \quad (9)$$

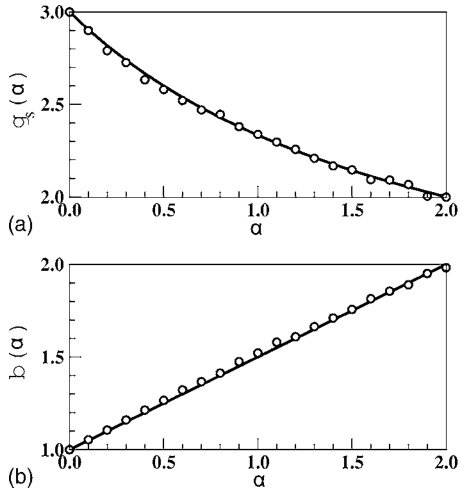

FIG. 5. Verification of the formulas for (a) $\gamma_s(\alpha)=1+4/(2+\alpha)$ and (b) $\beta(\alpha)=1+\alpha/2$ (solid lines) with the numerical results (circles) for $N=2^{14}$.

Since the serial number i and the strength s are both nodal quantities they are interdependent and their probability distributions obey $P(s,t)ds=-P(i,t)di$; the negative sign is because as i increases s decreases. Using $P(i,t)=1/t$ we get the time-dependent probability distribution of the nodal strengths as

$$P(s,t) = -\frac{1}{t}\frac{di}{ds} \propto (t^{1/2-\alpha/4}+c)^{4/(2+\alpha)}\left(\frac{4}{2+\alpha}s^{-1-4/(2+\alpha)}\right) \propto t^{-2\alpha/(2+\alpha)}s^{-1-4/(2+\alpha)}. \qquad (10)$$

Therefore $\gamma_s(\alpha)=1+4/(2+\alpha)$ and using the general relation $\gamma_s=\gamma_k/\beta+1-1/\beta$ [8] and $\gamma_k=3$ we get

$$\beta(\alpha) = 1 + \alpha/2. \qquad (11)$$

The exponent of the distribution of $w_{ij}=\ell_{ij}^{\alpha}$ varies as $\gamma_w(\alpha)=1+2/\alpha$. A number of checks have been done to verify these results. For a large network with $N=2^{14}$ the strength distribution and its average are calculated for 21 different values of α equally spaced between 0 and 2. Results are found to be quite consistent with the above formulas for $\gamma_s(\alpha)$ and $\beta(\alpha)$ (Fig. 5).

Therefore the message is that a nontrivial value of $\beta>1$ indeed can be obtained in a weighted network when the link weights decrease systematically as time elapses. In the above model the link weight decreases as $(it)^{-\alpha/4}$. This is consistent with airport network data where early airports in big cities like London, Paris, etc., have very high strengths. The fact that these early airports have survived over more than a century implies that they are connected with strong links of high passenger traffic as well as long-distance links. Therefore if the early links have maximal weights and if they decrease inversely as a power of time, that can result in $\beta>1$.

A simple way to check this idea is to study the generalized BA model itself, starting from a single link. The weights of the links are assigned by hand: A link which becomes a part of the network at time t carries a weight $t^{-\alpha}$. Then if the degree of the largest hub $\langle k_{max}\rangle$ grows as t^{μ} ($\mu=1/2$ for the BA model) it can be shown that, for $\alpha<\mu$, $\beta(\alpha)=1$, and for $\alpha>\mu$, $\beta(\alpha)=\alpha/\mu$. The weight distribution exponent varies with α as $\gamma_w(\alpha)=1+1/\alpha$. We observe that these results are very consistent with the results in [12]. In this model each link comes in with a constant weight and at the same time m' randomly selected links gain additional weights w_0 at each time step. As a result old links increase their weights as time increases. On simulating the Bianconi model network of size $N=2^{14}$ we calculated that the average weight of a link introduced at time i decreases as $i^{-\alpha}$, which is exactly the result of [12] that the weight of a link increases with time as $(t/i)^{\alpha}$. In comparison to the conclusion of [13] that a global reorganization of weights (as in [13,12]) yields $\beta>1$, we claim that ultimately the global reorganization or any other dynamics has to ensure a power law decay of the link weights like $t^{-\alpha}$ to achieve $\beta>1$. A similar conclusion has been drawn in [12].

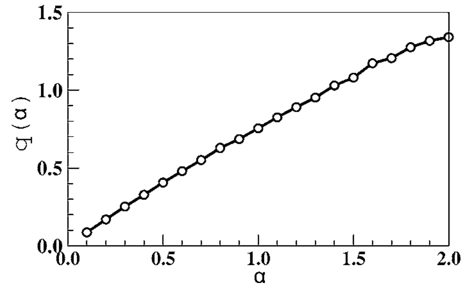

FIG. 6. Variation of the exponent $\theta(\alpha)$ with α for $N=2^{14}$.

To study the correlation between the nodal degrees and the link weights Barrat *et al.* [8] calculated how the product of the degrees k_ik_j of the two end points of a link varies with the link weight w_{ij},

$$\langle w_{ij}\rangle \propto (k_ik_j)^{\theta(\alpha)}. \qquad (12)$$

A direct calculation of $\langle w_{ij}\rangle$ averaged over different configurations generated for different values of α for our spatial network shows that the variation is almost linear (Fig. 6).

We further observe that our model can be generalized when the new node selects one of the t links randomly with a probability $d_n^{\delta}/\Sigma_{n=1}^{t}d_n^{\delta}$ where δ is a continuously tunable parameter. Therefore in the limit of $\delta\to-\infty$ we retrieve the above model. On the other hand for $\delta=0$ the underlying Euclidean space is switched off and the networks are generated in a much simpler way. To add the $(t+1)$th link, a link j is selected out of t links with uniform probability $1/t$. The new $(t+2)$th node is then randomly connected to one end of j with probability $1/2$. Therefore a node of degree k has the probability $k/(2t)$ to be linked with the new node, which clearly satisfies the requirement of linear attachment probability of the BA model. Therefore we claim that the algorithm where the new node selects one of the links with uniform probability and gets connected to its one end with probability $1/2$ is exactly the BA algorithm. Computationally there is a lot of advantage; it takes a CPU time linearly proportional to N as already observed in [2,17,18]. The network so generated clearly has a tree structure. However, loops can also be generated easily by connecting each new node with m distinct links, each link being attached with the same rule. The resulting network is exactly the BA network with m outgoing links from each node.

Finally it is observed that the case of the nonlinear attachment rule in BA model when the probability of attachment varies as k^{ϵ} cannot be obtained by generalizing this model. Let the two end nodes of the randomly selected link have degrees k_1 and k_2; then depending on their degrees one of them is selected with the following probabilities:

$$p(k_1) = k_1^{\epsilon}/(k_1^{\epsilon} + k_2^{\epsilon}) \quad \text{and} \quad p(k_2) = k_2^{\epsilon}/(k_1^{\epsilon} + k_2^{\epsilon}) \qquad (13)$$

with $p(k_1)+p(k_2)=1$. The BA model corresponds to $\epsilon=0$. However, on reducing ϵ gradually through its negative values, the node with the smaller degree gets more preference to be connected. Therefore in the limit of $\epsilon \to -\infty$ the node with smaller degree is always selected with probability 1. Even in this limit the network has a branched structure because if the two end nodes have equal degrees, any one of them gets connected with the new node with probability 1/2. On the other hand when $\epsilon > 0$, the node with a larger degree gets more weight and in the limit of $\epsilon \to \infty$ the end node with larger degree always gets the new connection, resulting in a starlike structure. When ϵ decreases below zero, the degree distribution becomes stretched-exponential-like, $P(k, \epsilon < 0) \sim \exp(-ak^{b(\epsilon)})$ where the exponent $b(\epsilon)$ has a continuous variation with ϵ, expected to reach 1 as $\epsilon \to -\infty$ and 0 when $\epsilon \to 0$. Our numerical estimates for $b(\epsilon)$ for different values of ϵ fit very well to the form $b(\epsilon)=a_0(-\epsilon)^{\nu}-b_0$ where the constants are estimated to be $a_0 \approx 1.19$, $\nu \approx 0.14$, and $b_0 \approx 0.85$.

To summarize, we argue that in the growing Internet as well as in the airport network it is more likely that the new nodes get their connections in the local neighborhood. Indeed, a spatial scale-free network is grown using the criterion of the local selection rule. This network shows a nonlinear dependence of the nodal strength on the degree. We conjecture that, irrespective of what the dynamics may be, the nonlinearity is the result of a power law decay of the link weight with time. When the same network is constructed without the underlying Euclidean space it gives a very efficient algorithm to generate the BA network.

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